Obtain the fixed point of nonlinear equations through the Whale Optimization Algorithm

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ABSTRACT

In this paper, we introduce a new iterative method to finding the fixed point of a nonlinear function. In fact, we want to offer a new way to obtain the fixed point of various functions using the Whale Optimization Algorithm. This method is new and very efficient for solving a nonlinear equation. We explain this method with four benchmark functions and compare results with others methods, such as ALO, MVO, SSA, SCA, GWO.

KEYWORDS: meta-heuristic algorithms, Fixed point problems, The Whale Optimization Algorithm.

1 Introduction

Obtaining the roots of equations, especially nonlinear equations, is one of the most important topics in engineering and basic sciences. For this sake, many researchers have checked this problem for some years [34,42].

The Bisection method is one of the most important methods in numerical calculations to find the root of a continuous function, which we know has a different sign at two points. This method is one of the simplest ways to find the root of a function in numerical calculations.

Meta-heuristic or meta-heuristic or meta-heuristic algorithms are a type of random algorithms that are used to find the optimal answer. Optimization methods and algorithms are divided into two categories: exact algorithms and approximate algorithms.

Well-known population- based meta-heuristic algorithms include evolutionary algorithms (genetic algorithm) [2], ant colony optimization (ACO) [3, 4], bee colony(BC) [5], particle

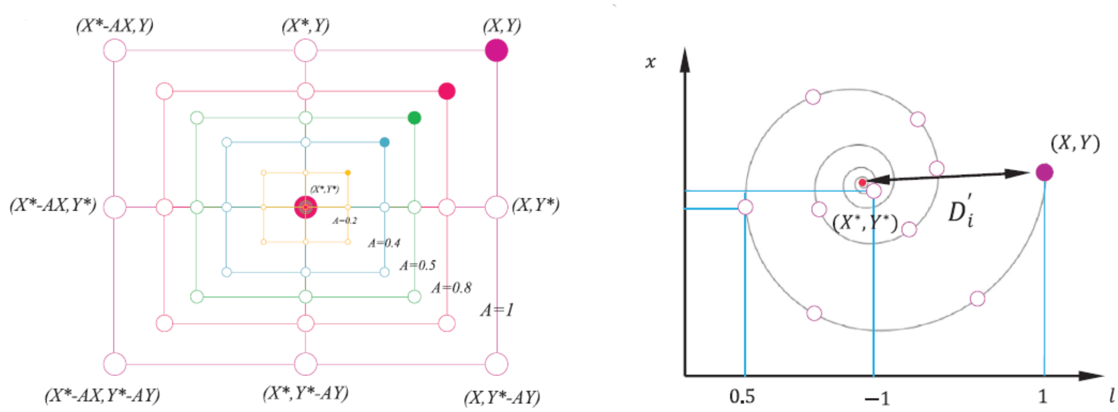
swarm optimization method (PSO) [6], forest optimization algorithm (FO) [7], Battle royale optimization algorithm (BRO) [8], runner- root algorithm(RRA) [9] , intelligent water drops algorithm (IWD) [10], Artificial Bee Colony algorithm(ABC) [11, 12], Firefly Algorithm(FA) [13] , Differential evolution (DE) algorithms [14], biogeography based optimization (BBO) algorithm [15].
In recent years, new meta-heuristic algorithms have been developed with respect to living organisms in nature (inspired by nature), the most famous of which are the Gray Wolf Optimization Algorithm(GWO) [16], the Dragonfly algorithm (DA) [17], the Flower Pollination Optimization Algorithm (FPA) [18], Whale optimization Algorithm (WOA) [19], Grasshopper Optimisation Algorithm (GOA) [20], social spider algorithm (SSA) [21], Sine Cosine Algorithm (SCA) [22], Multi-Verse Optimizer algorithm (MVO) [23], Moth-flame optimization algorithm (MFO) [24], Ant Lion Optimizer algorithm (ALO) [25], Emperor Penguins Colony algorithm [26] and so on [1,44], [27-33].
In this paper, we introduce a novel iterative method that obtain the fixed point of various functions using the Whale Optimization Algorithm.
In Sect. 2, the Whale Optimization Algorithm is explained and fixed point problem is illustrated. Also suggested method illustrated in Sect.3. Section 4 measures the resolution of the offered method by different methods on several functions. Also, the result is available at Sect. 5.

2 Preliminaries

In the present section, the Whale Optimization Algorithm is explained and fixed point problem is illustrated.

2.1 The Whale Optimization Algorithm (WOA)

The Whale or Whale Optimization Algorithm (WOA) is the subject of this section. This algorithm was presented by Seyed ali Mir jalili in 2016 in The Whale Optimization Algorithm in the journal Advances in Engineering Software at Elsevier.
Metaphysical optimization algorithms are becoming more and more popular in engineering applications. Because they rely on relatively simple concepts in the first place and are easy to implement. Second, they do not need gradient information. Third, they can circumvent the risk of local optimization and are ultimately used in a wide range of issues covering different disciplines. The following is an introduction to how the WOA optimization algorithm works.



(a) (b)

Figure 1: Bubble-net search mechanism implemented in WOA (X∗ is the best solution obtained so far): (a) shrinking encircling mechanism and (b) spiral updating position

One of the largest mammals in the world is the whale or whale. Among the 7 most famous whales in the world, the humpback whale is the humpback whale. An adult humpback whale is about the size of a school bus. Favorite prey is whales, krill and small fish groups. The most interesting thing about humpback whales is their special hunting method. This exploratory behavior is known as the Bubble-net feeding method.
Humpback whales prefer to hunt a group of krill or small fish near the surface of the water. It has been observed that this exploration and hunting is done by creating index bubbles along a circle or paths. The WOA algorithm is one of the nature-inspired and population-based optimization algorithms that can be used in various fields.
The WOA wall algorithm is performed in three steps or three phases as follows:
1- Siege hunting
2- Operation phase: The method of attacking the net bubble
3- Exploration stage: hunting search
Whales can identify hunting grounds and surround them. Since the optimal design location in the search space is not known by comparison, the algorithm assumes that the best candidate solution at the moment is target hunting or close to optimal. Once the best search engine is identified, other search agents try to update their location to the best search engine. This behavior is expressed through relationships (2.1) and (2.2) of the article:

$$\vec{D} = |\vec{C}.\vec{X^*}(t) - \vec{X}(t)| \quad (2.1)$$

$$\vec{X}(t+1) = \vec{X^*}(t) - \vec{A} \cdot \vec{D} \quad (2.2)$$

Where t denotes the current iteration, A and C are the coefficient vectors, X* the location vector is the best solution obtained now, and X is the location vector. It should be noted that if there is a better solution, X* should be updated in each iteration. Vectors A and C are calculated as follows:

$$\vec{A} = 2\vec{a} \cdot \vec{r} - \vec{a} \tag{2.3}$$

$$\vec{C} = 2 \cdot \vec{r} \tag{2.4}$$

Where a decreases linearly from 2 to 0 during the iterations (in both exploration and extraction phases) and r is a random vector at a distance of 0 to 1.
To mathematically model the bubble behavior of net walls, two methods have been designed:
1- Contractile blocking mechanism: This behavior is achieved by increasing the value of a in relation (2.3). The oscillation range A is reduced by a. In other words, A is a random value in the distance a to -a and a decreases from 2 to 0 during repetitions. By selecting random values of A at intervals of 1 to -1, the new location of the search agent can be defined anywhere between the principal location of the agent and the location of the current best agent.
2- Spiral updating location: This method first calculates the distance between the wall located in the X* and Y coordinates of the bait in X* and Y*. A spiral equation is created between the position of the whale and the prey to mimic the spiral motion of the humpback whale:

$$\vec{X}(t+1) = \overrightarrow{D'} \cdot e^{bl} \cdot \cos(2\pi l) + \overrightarrow{X^*}(t) \tag{2.5}$$

The following figure shows the pseudo-code of the above algorithm.

```
Initialize the whales population Xi (i = 1, 2, ..., n)
Calculate the fitness of each search agent
X*=the best search agent
while (t < maximum number of iterations)
    for each search agent
    Update a, A, C, l, and p
        if1 (p<0.5)
            if2 (|A| < 1)
                Update the position of the current search agent by the Eq. (2.1)
            else if2 (|A| ≥ 1)
                Select a random search agent (X_rand)
                Update the position of the current search agent by the Eq. (2.8)
            end if2
        else if1 (p≥0.5)
                Update the position of the current search by the Eq. (2.5)
        end if1
    end for
    Check if any search agent goes beyond the search space and amend it
    Calculate the fitness of each search agent
    Update X* if there is a better solution
    t=t+1
end while
return X*
```

Figure 2: Pseudo-code of the WOA algorithm

The flowchart of the Whale Optimization Algorithm is as follows:

Figure 3: Flowchart of the Wall Optimization Algorithm

2.2 Definition the fixed point

In mathematics, a fixed point (invariant point) of a function isa point that is mapped to itself by the function. In other word, a number c is a fixed point for a given function g if g(c) = c. A set of fixed points is sometimes called a fixed set. An iterative method for solving equation g(x) = x is the recursive relation xi+1= g(xi), i = 0, 1, 2, . . ., with some initial guess x0. The algorithm stops when one of the following stopping criterion is met:
•D1: total number of iterations is N, for some N, fixed a priori.
•D2: |xi+1− xi| < € for some €, fixed a priori.
This procedure is shown in figure 4.

Repeat

1. Give the transcendental equation $x = g(x)$,
2. Give initial guess in interval $[a, \quad b]$,
3. Do $x_{i+1} = g(x_i)$

until (D_1 or D_2 is true)

Figure 4: Fixed Point iteration scheme

3 The Whale Optimization Algorithm for solving fixed point of functions

At present part, we present one modern repetitious procedure to gain the solution estimation of a fixed point question as g(x) = x. We describe a function f(x) = g(x) - x. Accordingly the question of discovering the fixed points of g(x) is decreased to discovering the roots of f(x). We subsequent describe a function h(x) =|f(x)|. The question of discovering the roots of f(x) is better decreased to discovering an x that minimizes h(x). The opinion where is that for obtaining the half point of the distance I to begin with one volunteer answer, WOA algorithm is utilized to impute one superior estimation and determined a distance Ik = [ak , bk] one volunteer solution xk is calculated utilizing the WOA algorithm. If f(xk) = 0 we are accomplished, again calculate one modern distance Ik+1 into Ik pertaining against whether f(xk). f(ak) < 0 or f(xk). f(bk) < 0.

4 Implement methods on various functions

In this section, we illustrate our algorithm with some examples and compare the results with other evolutionary optimization algorithms such as ALO, MVO, SSA, SCA, GWO.

4.1 Introducing different functions

Introducing different functions

$$g_1(x): \frac{x^3}{150} - 2\sin(x) = x \quad ; \quad x \in [-10,+10] \quad ; \quad fp = 0$$

$$g_2(x): 3x^4 + 4\cos(x) - 4 = x \quad ; \quad x \in [-15,+15] \quad ; \quad fp = 0$$

$$g_3(x): x\cos(x) - \frac{x^2}{3}\sin(x) + 2\pi = x \quad ; \quad x \in [-2\pi,+2\pi] \quad ; \quad fp = \pi$$

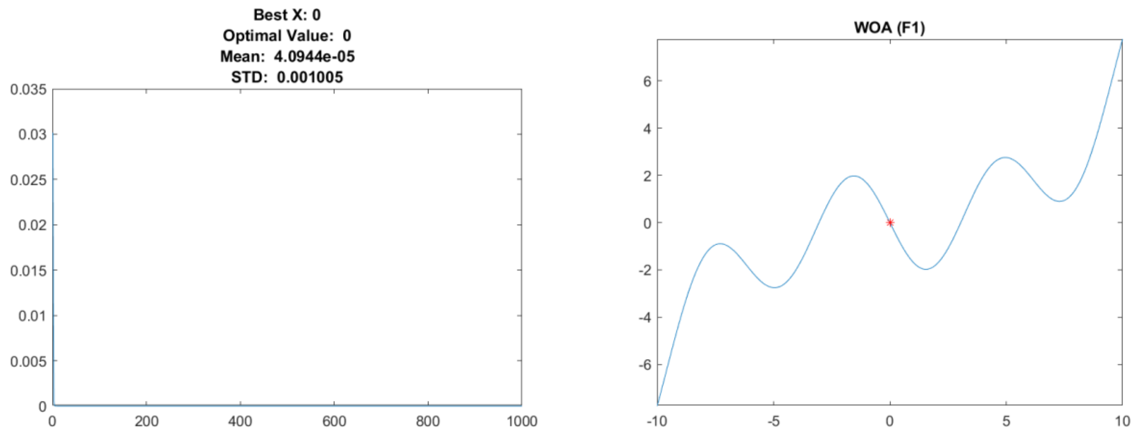
$$g_4(x): e + 4 - e^{\cos(2\pi x)} - e^{0.5\sqrt{x^2}} = x \quad ; \quad x \in [1,+10] \quad ; \quad fp = 4$$

Results for the three functions are shown in Table 1 and their diagrams are also shown in Figures 5-8. Figures 5 to 8 show Diagram of the recovery process of the g1 to g4 functions by the WOA algorithm in (a) and diagram of the finding of the fixed point of the g1 to g4 functions by the WOA algorithm in (b).

Table 1 The Comparative results obtained for each function by ALO, MVO, SSA, SCA, GWO and WOA algorithms

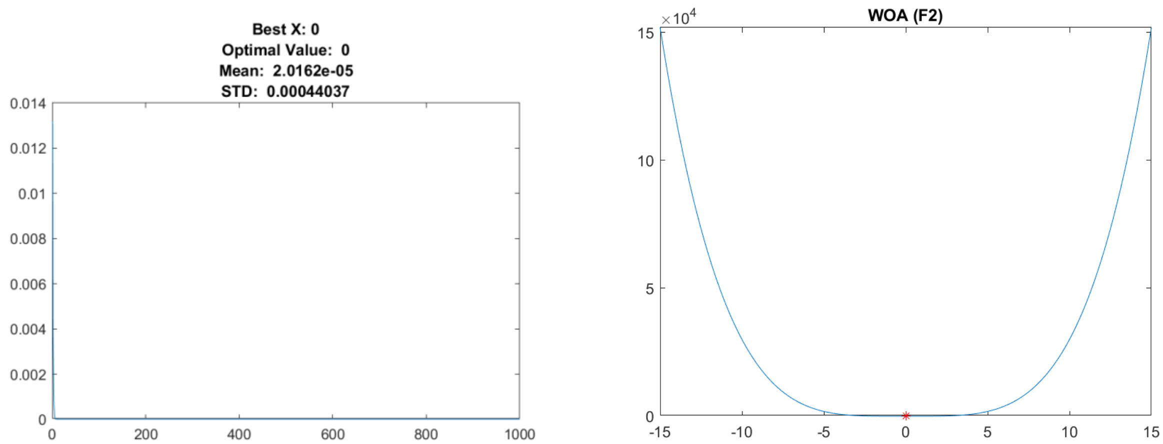
algorithm	Components	g1(x)	g2(x)	g3(x)	g4(x)
ALO	error	9.76E-12	1.35E-09	3.42E-10	7.46E-10
	X_best	-3.3E-12	-1.3E-09	3.141593	2.600822
	mean(e)	3.58E-05	4.65E-05	6.73E-06	5.37E-05
	std(e)	0.000547	0.000172	2.75E-05	0.000524
MVO	error	2.48E-08	2.19E-07	1.78E-08	4.15E-06
	X_best	-8.3E-09	-2.2E-07	3.141593	1.898763
	mean(e)	0.000101	0.000333	8.25E-05	0.000625
	std(e)	0.001331	0.002159	0.000394	0.001687

SCA	error	2.6E-129	9.8E-127	1.71E-07	2.14E-07
	X_best	8.8E-130	9.8E-127	2.772928	2.600822
	mean(e)	0.000211	1.34E-05	1.6E-05	6.73E-05
	std(e)	0.006502	0.00026	9.49E-05	0.000231
SSA	error	4.46E-10	4.99E-10	3.79E-11	8.89E-09
	X_best	-1.5E-10	0.97781	3.141593	2.600822
	mean(e)	3.32E-05	5.77E-05	2.75E-05	0.001547
	std(e)	0.000134	0.000718	0.000126	0.00548
GWO	error	0	0	1.21E-07	5.26E-08
	X_best	0	0	2.772928	2.174708
	mean(e)	3.85E-05	1.04E-05	1.33E-05	3.05E-05
	std(e)	0.000791	0.00026	7.05E-05	0.000257
WOA	error	0	0	4.62E-12	1.18E-11
	X_best	0	0	2.772928	2.600822
	mean(e)	4.09E-05	2.02E-05	6.04E-06	1.38E-05
	std(e)	0.001005	0.00044	0.000149	0.0002



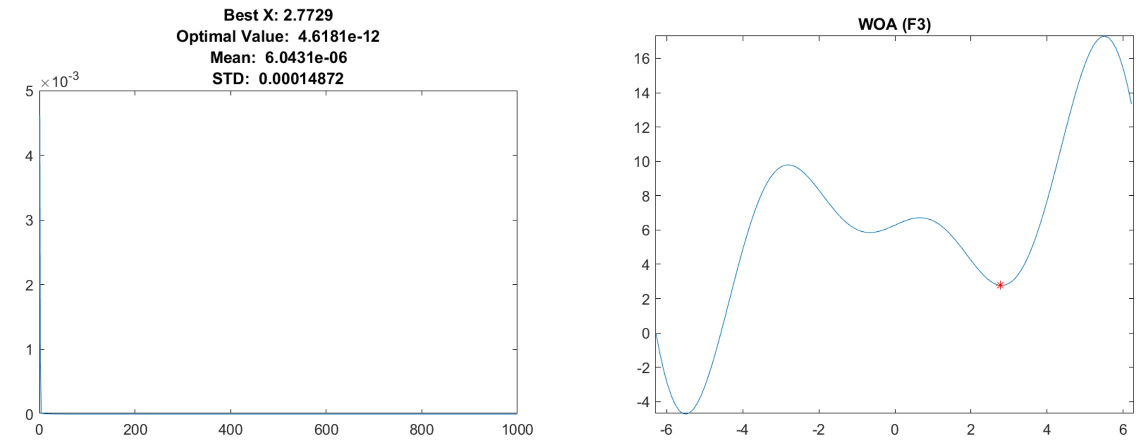
(a) (b)

Figure *5*: Diagram of the recovery process of the g1 function by the WOA algorithm in (a) and diagram of the finding of the fixed point of the g1 function by the WOA algorithm using the intersection of the diagram g1 (x) = x in (b)



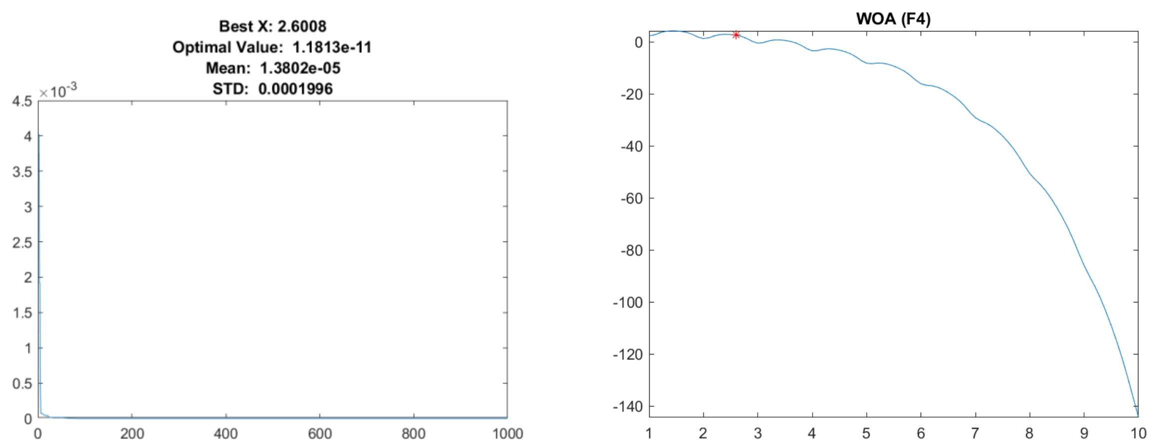
(a) (b)

Figure *6*: : Diagram of the recovery process of the g2 function by the WOA algorithm in (a) and diagram of the finding of the fixed point of the g2 function by the WOA algorithm using the intersection of the diagram g2 (x) = x in (b)



(a) (b)

Figure *7*: : Diagram of the recovery process of the g3 function by the WOA algorithm in (a) and diagram of the finding of the fixed point of the g3 function by the WOA algorithm using the intersection of the diagram g3 (x) = x in (b)



(a) (b)

Figure ***8***: : Diagram of the recovery process of the g4 function by the WOA algorithm in (a) and diagram of the finding of the fixed point of the g4 function by the WOA algorithm using the intersection of the diagram g4 (x) = x in (b)

5 conclusion

In this paper, we introduce a novel iterative method for finding a fixed point of a function g in a real interval $[a, b] \subseteq R$ by using the Whale Optimization Algorithm. If the function g is hard, it is sometimes difficult to determine suitable initial value close to the location of a fixed point. Derivative method (find the derivative of $g(x) - x$ and find its root) is also sometimes not useful for various reasons like the derivative may not exist, the derivative is hard to compute or finding the root of a derivative itself may be difficult. WOA algorithm helps in finding a good initial value and the proposed method does away with the need to compute the derivative. Our proposed algorithm is easy to use and reliable. As comparison with other algorithm shows, the accuracy of our proposed method also is good.

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